THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2050B Mathematical Analysis I (Fall 2016)
Suggested Solutions to Homework 5

Let $A \subseteq \mathbb{R}$ be nonempty, $f : A \to \mathbb{R}$, and $c \in \mathbb{R}$ be a cluster point of A.

1. Suppose $f(x_n)$ converges in $\mathbb{R}$ whenever (x_n) is a sequence in $A \backslash \{c\}$ converging to c. Show that there exists $l \in \mathbb{R}$ such that $f(x_n)$ converges to l whenever (x_n) is a sequence in $A \backslash \{c\}$ converging to c. Hence, by virtue of definition of limits for functions, show that $\lim_{x \to c} f(x) = l$.

Proof.

- We first prove that for any sequences (x_n), (y_n) with $(x_n), (y_n) \subseteq A \backslash \{c\}$, $(x_n) \to c, (y_n) \to c$, we have

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f(y_n)$$

Indeed, let (x_n), (y_n) as above. By assumption, $f(x_n) \to l$, $f(y_n) \to l'$ for some $l, l' \in \mathbb{R}$. We would like to show that $l = l'$. To this end, we construct a new sequence (z_n) as:

$$(z_n) := (x_1, y_1, x_2, y_2, \dots, x_n, y_n, \dots)$$

Then we have $(z_n) \subseteq A \backslash \{c\}$, and that $(z_n) \to c$. By assumption again, we have $f(z_n) \to l''$ for some $l'' \in \mathbb{R}$. But by construction, (x_n) is a subsequence of (z_n), whence $f(x_n)$ is a subsequence of $f(z_n)$. Since $f(z_n) \to l''$, we must have $f(x_n) \to l''$. Hence $l = l''$.

Similarly, $l' = l''$. This gives $l = l'$.

- We pick a fixed sequence $(a_n) \subseteq A \backslash \{c\}$ converging to c. The existence of such sequence is guaranteed by the assumption that c is a cluster point of A. Then there exists $l \in \mathbb{R}$ with $\lim_{n\to\infty} f(a_n) = l$. By the above claim, $\lim_{n\to\infty} f(x_n) = l$ for any $(x_n) \subseteq A \backslash \{c\}$ converging to c.

- Lastly we show that $\lim_{x\to c} f(x) = l$.

Suppose not. Then there exists $\epsilon_0 > 0$ such that for any $\delta > 0$, there exists $x \in A$ with $0 < |x - c| < \delta$ such that $|f(x) - l| \geq \epsilon_0$. In particular, for each $n \in \mathbb{N}$, we take $\delta_n := \frac{1}{n} > 0$ (or $\frac{689}{n^{1997}}$, $\frac{1}{2047^n}$ if you like, as long as it converges to 0 and strictly positive), and $x_n \in A$ be such that $0 < |x_n - c| < \delta_n$ and $|f(x_n) - l| \geq \epsilon_0$. Since $\delta_n \to 0$, by squeeze law, we have $x_n \to c$. In this way we obtain a sequence $(x_n) \subseteq A \backslash \{c\}$ with $x_n \to c$. By what we have proved just now, $\lim_{n\to\infty} f(x_n) = l$. Hence for $\epsilon := \epsilon_0 > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $|f(x_n) - l| < \epsilon_0$. This is a contradiction to our choice of x_n. Therefore the contrapositive is true, i.e. $\lim_{x\to c} f(x) = l$.

□

2. Suppose that for any $\epsilon > 0$ there exists $\delta > 0$ such that whenever $x, x' \in A\backslash\{c\}$ with $|x - c| < \delta$, $|x' - c| < \delta$, we have

$$|f(x) - f(x')| < \epsilon.$$

Show that $\lim_{x\to c} f(x)$ exists.

Proof. We will use the criterion in Question 1, i.e. we want to show that $f(x_n)$ converges in $\mathbb{R}$ whenever (x_n) is a sequence in $A\backslash\{c\}$ converging to c.

Let (x_n) be a sequence in $A\backslash\{c\}$ converging to c. Let $\epsilon > 0$. By assumption, there exists $\delta > 0$ such that whenever $x, x' \in A\backslash\{c\}$ with $|x - c| < \delta$, $|x' - c| < \delta$, we have

$$|f(x) - f(x')| < \epsilon. \qquad (*)$$

Since $(x_n) \to c$, there exists $N \in \mathbb{N}$ such that for $n \geq N$, $|x_n - c| < \delta$, and that $x_n \neq c$. Then if $n, m \geq N$, we have $0 < |x_n - c| < \delta$, $0 < |x_m - c| < \delta$. By (*), $|f(x_n) - f(x_m)| < \epsilon$. This shows that $f(x_n)$ is a Cauchy sequence. By Cauchy criterion of sequences, $f(x_n)$ converges in $\mathbb{R}$. By Question 1, there exists $l \in \mathbb{R}$ such that $\lim_{x\to c} f(x) = l$. □

3. Let (x_n) be a sequence of real numbers. Define

$$s_n := x_1 + x_2 + \cdots + x_n$$

$$s'_n := |x_1| + |x_2| + \cdots + |x_n|$$

Show that if (s'_n) converges to a real number, then so is (s_n).

Proof. We will show that (s_n) is a Cauchy sequence in $\mathbb{R}$. Let $\epsilon > 0$. Since (s'_n) is convergent, (s'_n) is Cauchy. Then there exists $N \in \mathbb{N}$ such that for $n > m \geq N$, we have $|s'_n - s'_m| < \epsilon$. Thus

$$\big||x_{m+1}| + |x_{m+2}| + \cdots + |x_n|\big| < \epsilon$$

By triangle inequality, we have:

$$|s_n - s_m| = |x_{m+1} + x_{m+2} + \cdots + x_n| \leq |x_{m+1}| + |x_{m+2}| + \cdots + |x_n| < \epsilon$$

This shows that (s_n) is Cauchy in $\mathbb{R}$. By Cauchy criterion, (s_n) is convergent in $\mathbb{R}$. □

4. Let (x_n) be a sequence which is not Cauchy. Show that there is an $\epsilon > 0$ such that:

   (a) For any $N \in \mathbb{N}$ there exists $N' \in \mathbb{N}$ with $N' > N$ such that $|x_N - x_{N'}| \geq \epsilon$.

   (b) There is a subsequence (x_{n_k}) of (x_n) such that $|x_{n_k} - x_{n_{k+1}}| \geq \epsilon$ for all $k \in \mathbb{N}$.

Proof. (a) We will prove by contradiction. Suppose for any $\epsilon > 0$, there is $N \in \mathbb{N}$ such that for any $n \in \mathbb{N}$ with $n > N$, we have $|x_N - x_n| < \epsilon$. Taking $m, n > N$, we have

$$|x_m - x_n| \leq |x_m - x_N| + |x_n - x_N| < 2\epsilon$$

This shows that (x_n) is Cauchy, which is a contradiction. Therefore there is an $\epsilon > 0$ such that for any $N \in \mathbb{N}$ there exists $N' \in \mathbb{N}$ with $N' > N$ such that $|x_N - x_{N'}| \geq \epsilon$.

(b) We will construct (x_{n_k}) inductively: Let $\epsilon > 0$ be as in (a).
Let $N = 1$. Then there exists $N_1 \in \mathbb{N}$ with $N_1 > 1$ such that

$$|x_1 - x_{N_1}| \geq \epsilon.$$

Let $N = N_1 \in \mathbb{N}$. Then there exists $N_2 \in \mathbb{N}$ with $N_2 > N_1$ such that

$$|x_{N_2} - x_{N_1}| \geq \epsilon.$$

Let $N = N_2 \in \mathbb{N}$. Then there exists $N_3 \in \mathbb{N}$ with $N_3 > N_2$ such that

$$|x_{N_3} - x_{N_2}| \geq \epsilon.$$

Inductively, we obtain in this fashion a subsequence (x_{n_k}) such that

$$|x_{N_{k+1}} - x_{N_k}| \geq \epsilon,$$

for any $k \in \mathbb{N}$. Therefore the sequence (x_{N_k}) is what we want (After changing the notations)

□